Algorithms – Kalman Filter

State vectors $\mathbf{x}$ at two subsequent epochs are related to each other by the following linear equation:

$$\mathbf{x}(n) = \mathbf{\Phi}\, \mathbf{x}(n-1) + \mathbf{\Gamma}\, \mathbf{w}(n) \ ,$$

where Φ and Γ are known matrices and *white noise* $\mathbf{w}(n)$ is a random vector with the following statistical properties:

$$\begin{aligned} E(\mathbf{w}) &= \mathbf{0} \\ E(\mathbf{w}(n)\, \mathbf{w}^T(m)) &= \mathbf{0} \ \text{ for } m \neq n \\ E(\mathbf{w}(n)\, \mathbf{w}^\mathbf{T}(n)) &= \mathbf{Q}_s(n) \ . \end{aligned}$$

Observations $\mathbf{l}(n)$ and the state vector $\mathbf{x}(n)$ are related to each other by the linearized *observation equations* of form

$$\mathbf{l}(n) = \mathbf{A}\, \mathbf{x}(n) + \mathbf{v}(n) \ ,$$

where $\mathbf{A}$ is a known matrix (the so-called *first-design matrix*) and $\mathbf{v}(n)$ is a vector of random errors with the following properties:

$$\begin{aligned} E(\mathbf{v}) &= \mathbf{0} \\ E(\mathbf{v}(n)\, \mathbf{v}^T(m)) &= \mathbf{0} \ \text{ for } m \neq n \\ E(\mathbf{v}(n)\, \mathbf{v}^\mathbf{T}(n)) &= \mathbf{Q}_l(n) \ . \end{aligned}$$

Classical KF Form

Minimum Mean Square Error (MMSE) estimate $\widehat{\mathbf{x}}(n)$ of vector $\mathbf{x}(n)$ meets the condition $E\left((\mathbf{x}-\widehat{\mathbf{x}})(\mathbf{x}-\widehat{\mathbf{x}})^T\right) = \min$ and is given by

$$\widehat{\mathbf{x}}^-(n) = \mathbf{\Phi}\widehat{\mathbf{x}}(n-1) \tag{1a}$$

$$\mathbf{Q}^-(n) = \mathbf{\Phi}\mathbf{Q}(n-1)\mathbf{\Phi}^T + \mathbf{\Gamma}\mathbf{Q}_s(n)\mathbf{\Gamma}^T \tag{1b}$$

$$\widehat{\mathbf{x}}(n) = \widehat{\mathbf{x}}^-(n) + \mathbf{K}\left(\mathbf{l} - \mathbf{A}\widehat{\mathbf{x}}(n-1)\right) \tag{2a}$$

$$\mathbf{Q}(n) = \mathbf{Q}^-(n) - \mathbf{K}\mathbf{A}\mathbf{Q}^-(n)\ , \tag{2b}$$

where

$$\mathbf{K} = \mathbf{Q}^-(n)\mathbf{A}^T\mathbf{H}^{-1}, \quad \mathbf{H} = \mathbf{Q}_l(n) + \mathbf{A}\mathbf{Q}^-(n)\mathbf{A}^T\ .$$

Equations (1) are called *prediction*, equations (2) are called *update* step of Kalman filter.

Square-Root Filter

Algorithms based on equations (1) and (2) may suffer from numerical instabilities that are primarily caused by the subtraction in (2b). This deficiency may be overcome by the so-called *square-root* formulation of the Kalman filter that is based on the so-called *QR-Decomposition*. Assuming the Cholesky decompositions

$$\mathbf{Q}(n) = \mathbf{S}^T\mathbf{S}, \quad \mathbf{Q}_l(n) = \mathbf{S}_l^T\mathbf{S}_l, \quad \mathbf{Q}^-(n) = \mathbf{S}^{-T}\mathbf{S}^- \tag{3}$$

we can create the following block matrix and its QR-Decomposition:

$$\begin{pmatrix} \mathbf{S}_l & \mathbf{0} \\ \mathbf{S}^-\mathbf{A}^T & \mathbf{S}^- \end{pmatrix} = N \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix} . \tag{4}$$

It can be easily verified that

$$\begin{aligned} \mathbf{H} &= \mathbf{X}^T\mathbf{X} \\ \mathbf{K}^T &= \mathbf{X}^{-1}\mathbf{Y} \\ \mathbf{S} &= \mathbf{Z} \\ \mathbf{Q}(n) &= \mathbf{Z}^T\mathbf{Z} . \end{aligned}$$

State vector $\widehat{\mathbf{x}}(n)$ is computed in a usual way using the equation (2a).